Physics 523, Quantum Field Theory II

Homework 5

Due Wednesday, 11th February 2004

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The Electron Self-Energy

1. We are to verify the equation,

$$\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{1}{[\ell^2-\Delta]^2}-\frac{1}{[\ell^2-\Delta_\Lambda]^2}\right)=\frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right).$$

To evaluate this, we will consider differentiation of the integral with respect to both Δ and Δ_Λ, considering them as separate, independent variables. Because the integration will commute with these derivatives, we may use our results of to see

$$\begin{aligned}\frac{d}{d\Delta}\frac{d}{d\Delta_\Lambda}\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{1}{[\ell^2-\Delta]^2}-\frac{1}{[\ell^2-\Delta_\Lambda]^2}\right) &= \int \frac{d^4\ell}{(2\pi)^4}\frac{d}{d\Delta}\frac{d}{d\Delta_\Lambda}\left(\frac{1}{[\ell^2-\Delta]^2}-\frac{1}{[\ell^2-\Delta_\Lambda]^2}\right),\\
&= \int \frac{d^4\ell}{(2\pi)^4}\left(\frac{d}{d\Delta}\frac{1}{[\ell^2-\Delta]^2}-\frac{d}{d\Delta_\Lambda}\frac{1}{[\ell^2-\Delta_\Lambda]^2}\right),\\
&= 2\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{1}{[\ell^2-\Delta]^2}-\frac{1}{[\ell^2-\Delta_\Lambda]^2}\right),\\
&= 2\frac{-i}{(4\pi)^2}\frac{1}{2}\left(\frac{1}{\Delta}-\frac{1}{\Delta_\Lambda}\right),\\
&= \frac{i}{(4\pi)^2}\left(\frac{1}{\Delta_\Lambda}-\frac{1}{\Delta}\right),\\
&= \frac{i}{(4\pi)^2}\frac{d}{d\Delta}\frac{d}{d\Delta_\Lambda}\log\left(\frac{\Delta_\Lambda}{\Delta}\right).\end{aligned}$$

Because the differentiation clearly commutes with the constant factor, we have that

$$\therefore \int \frac{d^4\ell}{(2\pi)^4}\left(\frac{1}{[\ell^2-\Delta]^2}-\frac{1}{[\ell^2-\Delta_\Lambda]^2}\right)=\frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right). \tag{1.1}$$

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2. We are to find the roots of the simple quadratic,

$$(1-x)m_0^2+x\mu^2-x(1-x)p^2=x^2p^2-x(p^2+m_0^2-\mu^2)+m_0^2=0.$$

Invoking the quadratic formula, we see immediately that the roots are given by

$$\begin{aligned}x &= \frac{p^2+m_0^2-\mu^2\pm\sqrt{(p^2+m_0^2-\mu^2)^2-4p^2m_0^2}}{2p^2},\\
&= \frac{1}{2}+\frac{m_0^2}{2p^2}-\frac{\mu^2}{2p^2}\pm\frac{1}{2p^2}\sqrt{p^4-2p^2(m_0^2+\mu^2)+(m_0^2-\mu^2)^2},\\
\therefore x &= \frac{1}{2}+\frac{m_0^2}{2p^2}-\frac{\mu^2}{2p^2}\pm\frac{1}{2p^2}\sqrt{[p^2-(m_0+\mu)^2][p^2-(m_0-\mu)^2]}.\end{aligned} \tag{2.1}$$

3. We are to verify that when $p^2>(m_0^2+\mu^2)$ there is at least one real root of the equation where $x\in(0,1)$. First, we will show that the solutions are real. By checking the discriminant, we see that

$$[p^2-(m_0+\mu)^2][p^2-(m_0-\mu)^2]>[p^2-m_0^2-\mu^2][p^2-m_0^2-\mu^2+2m_0\mu]>1[1+2m_0\mu]>0.$$

Therefore the quadratic has only real roots. Now, let us show that the sum of the two solutions is positive. Noting that $\mu^2>0$, we have

$$x_1+x_2=1-\frac{m_0^2-\mu^2}{p^2}>1-\frac{m_0^2+\mu^2}{p^2}>0.$$

Therefore at least one of the two solutions must be positive. Lastly, we can show that the product of the two solutions is positive. This will guarantee that both solutions must be positive. By direct computation, we have

$$\begin{aligned}
x_1x_2 &= \frac{1}{4p^4}\left(\left(p^2+m_0^2-\mu^2\right)^2-\left(p^2-(m_0+\mu)^2\right)\left(p^2-(m_0-\mu)^2\right)\right),\\
&= \frac{1}{4p^4}\left(\left(p^2+m_0^2-\mu^2\right)^2-\left(p^2-m_0^2-\mu^2-2m_0\mu\right)\left(p^2-m_0^2-\mu^2+2m_0\mu\right)\right),\\
&> \frac{1}{4p^4}\left(\left(p^2+m_0^2-\mu^2\right)^2-\left(p^2+m_0^2-\mu^2-2m_0\mu\right)\left(p^2+m_0^2-\mu^2+2m_0\mu\right)\right),\\
&= \frac{1}{4p^4}\left(\left(p^2+m_0^2-\mu^2\right)^2-\left(p^2+m_0^2-\mu^2\right)^2+4m_0\mu\right),\\
&= \frac{m_0\mu}{p^4}>0.
\end{aligned}$$

Therefore, there are two real solutions to the equation. To show that a solution is confined to the interval $(0,1)$ we note that in the physically reasonable case where $\mu\to 0$, the x_2 solution becomes

$$\begin{aligned}
x_2 &= \frac{1}{2p^2}\left(p^2+m_0^2-\sqrt{[p^2-m_0^2][p^2-m_0^2]}\right),\\
&= \frac{1}{2p^2}\left(p^2+m_0^2-\mu^2-p^2-m_0^2\right),\\
&= \frac{m_0^2}{p^2}<1.
\end{aligned}$$

Therefore $x\in(0,1)$ is a real root of the quadratic equation of interest.

4. We are to show that $\delta F_1(0)+\delta Z_2=0$. To do this, we must first compute $\delta F_1(0)$. Let us recall the content of Peskin equation (6.47) while taking $q\to 0$,

$$\overline{u}(p')\delta\Gamma^\mu u(p)=4ie^2\int_0^1 dxdydz\delta^{(3)}(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\frac{\overline{u}(p')\left[\gamma^\mu\cdot\left(-\frac{1}{2}\ell^2+(1-4z+z^2)m^2\right)\right]u(p)}{[\ell^2-\Delta]^3}.$$

We see that this term is just proportional to the $\delta F_1(0)$ term in our expression for $\delta\Gamma^\mu$. To actually compute this integral, we will require Pauli-Villars regularization of the term proportional to ℓ^2. Also, we will use the fact that $\lim_{\Lambda\to\infty}\Delta_\Lambda = z\Lambda^2$. Now, invoking the results of homework 2, we have that

$$\begin{aligned}
\delta F_1(0) &= 4ie^2\int_0^1 dxdydz\delta^{(3)}(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\left[\left(-\frac{1}{2}\right)\left(\frac{\ell^2}{[\ell^2-\Delta]^3}-\frac{\ell^2}{[\ell^2-\Delta_\Lambda]^3}\right)+\frac{(1-4z+z^2)m^2}{[\ell^2-\Delta]^3}\right],\\
&= 4ie^2\int_0^1 dxdydz\delta^{(3)}(x+y+z-1)\left[\frac{-i}{2(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right)-\frac{i}{2(4\pi)^2}\frac{(1-4z+z^2)m^2}{\Delta}\right],\\
&= \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta^{(3)}(x+y+z-1)\left[\log\left(\frac{\Delta_\Lambda}{\Delta}\right)+\frac{(1-4z+z^2)m^2}{\Delta}\right],\\
&= \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta^{(3)}(x+y+z-1)\left[\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right)+\frac{(1-4z+z^2)m^2}{(1-z)^2m^2+z\mu^2}\right],\\
&= \frac{\alpha}{2\pi}\int_0^1 dz(1-z)\left[\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right)+\frac{(1-4z+z^2)m^2}{(1-z)^2m^2+z\mu^2}\right].
\end{aligned}$$

Quoting Peskin equation (7.31),

$$\delta Z_2=\frac{\alpha}{2\pi}\int_0^1 dz\left[-z\log\left(\frac{z\Lambda^2}{(1-z)^2+z\mu^2}\right)+\frac{2z(2-z)(1-z)m^2}{(1-z)^2m^2+z\mu^2}\right].$$

Therefore,

$$\begin{aligned}
\delta F_1(0)-\delta Z_2 &= \frac{\alpha}{2\pi}\int_0^1 dz\left[(1-2z)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right)+\frac{(1-z)(1-4z+z^2)m^2+2z(2-z)(1-z)m^2}{(1-z)^2m^2+z\mu^2}\right],\\
&= \frac{\alpha}{2\pi}\int_0^1 dz\left[(1-2z)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right)+\frac{m^2(z^3-z^2-z+1)}{(1-z)^2m^2+z\mu^2}\right].
\end{aligned}$$

To evaluate this integral, we will integrate the first part using integration by parts. Recall that, in general, $\left(\log \frac{f}{g}\right)' = \frac{f'}{f} - \frac{g'}{g} = \frac{f'g - g'f}{fg}$. Therefore, we may compute,

u		dv
$\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right)$	$\searrow^{+}$	$(1-2z)$
$\frac{m^2(1-z^2)}{z((1-z)^2m^2+z\mu^2)}$	$\xleftarrow{-}$	$z - z^2$

$$\int_0^1 dz(1-2z)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right),$$

$$\begin{aligned}
&= (z-z^2)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right)\Bigg|_0^1 - \int_0^1 dz\frac{m^2(1-z^2)(z-z^2)}{z\left((1-z)^2m^2+z\mu^2\right)},\\
&= 0 - \int_0^1 dz\frac{m^2(1-z^2)(z-z^2)}{z\left((1-z)^2m^2+z\mu^2\right)},
\end{aligned}$$

$$\therefore \int_0^1 dz(1-2z)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right) = -\int_0^1 dz\frac{m^2(z^4-z^3-z^2+z))}{z\left((1-z)^2m^2+z\mu^2\right)}.$$

Therefore, we readily see that

$$\begin{aligned}
\delta F_1(0) - \delta Z_2 &= \frac{\alpha}{2\pi}\int_0^1 dz\left[(1-2z)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right) + \frac{m^2(z^3-z^2-z+1)}{(1-z)^2m^2+z\mu^2}\right],\\
&= \frac{\alpha}{2\pi}\int_0^1 dz\left[\frac{m^2(z^4-z^3-z^2+z)}{z\left((1-z)^2m^2+z\mu^2\right)} + \frac{m^2(z^3-z^2-z+1)}{(1-z)^2m^2+z\mu^2}\right],\\
&= \frac{\alpha}{2\pi}\int_0^1 dz\left[\frac{m^2(z^4-z^3-z^2+z)}{z\left((1-z)^2m^2+z\mu^2\right)} + \frac{z}{z}\frac{m^2(z^3-z^2-z+1)}{(1-z)^2m^2+z\mu^2}\right],\\
&= \frac{\alpha}{2\pi}\int_0^1 dz\left[\frac{m^2(z^4-z^3-z^2+z)}{z\left((1-z)^2m^2+z\mu^2\right)} + \frac{m^2(z^4-z^3-z^2+z)}{z\left((1-z)^2m^2+z\mu^2\right)}\right],\\
&= 0.
\end{aligned}$$

$$\therefore \delta F_1(0) - \delta Z_2 = 0.$$

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